

The Dirichlet problem for singular fully nonlinear operators

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1 Introduction

In the last decade, fully-nonlinear elliptic equations have been extensively studied, both in the variational and the non variational setting. The concept of viscosity solution is particularly appropriate when considering non variational fully-non linear operators. It is interesting to recall that this concept relies deeply on a local comparison test and it is, hence, the perfect tool for studying comparison and maximum principles.

In a different contest of linear operators and strong solutions, Berestycki, Nirenberg and Varadhan in [1], used the maximum principle to define the concept of eigenvalue and to prove existence of solutions for Dirichlet problems when the boundary of the domain doesn't satisfy any regularity condition.

These considerations have lead to extend the concept of eigenvalue to the larger class of fully-nonlinear operators, using a definition which is analogous to the one given in [1] via the Maximum Principle, together with the concept of viscosity solutions.

We shall now make this more precise. Let Ω be a bounded domain of \mathbb{R}^N with a C^2 boundary, let S be the set of symmetric matrices $N \times N$, let $\alpha > -1$, we consider operators

$$F(x, \nabla u, D^2u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) |u|^\alpha u$$

with $F : \Omega \times \mathbb{R}^N \times S$ continuous, satisfying

$$(H1) \quad F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X), \forall t \in \mathbb{R}^*, \mu \in \mathbb{R}^+ \quad \forall x \in \Omega.$$

$$(H2) \quad \text{There exist } A \geq a > 0 \text{ such that for all } x \in \Omega, p \neq 0 \text{ and for all } (M, N) \in S^2, \\ N \geq 0$$

$$a|p|^\alpha \text{tr} N \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha \text{tr} N$$

and with b and c bounded and continuous.

The role of eigenvalue will be played by

$$\bar{\lambda} = \sup\{\lambda, \exists \varphi > 0 \text{ in } \Omega, F(x, \nabla \varphi, D^2 \varphi) + b(x) \cdot \nabla \varphi |\nabla \varphi|^\alpha + (c(x) + \lambda) |\varphi|^\alpha \varphi \leq 0 \text{ in } \Omega\}$$

and

$$\underline{\lambda} = \sup\{\lambda, \exists \varphi < 0 \text{ in } \Omega, F(x, \nabla \varphi, D^2 \varphi) + b(x) \cdot \nabla \varphi |\nabla \varphi|^\alpha + (c(x) + \lambda)|\varphi|^\alpha \varphi \geq 0 \text{ in } \Omega\}$$

Precisely it is possible to prove that there exist two non trivial functions $\phi_1 \geq 0$ and $\phi_2 \leq 0$ respectively solution of

$$\begin{cases} F(x, \nabla \phi_1, D^2 \phi_1) + b(x) \cdot \nabla \phi_1 |\nabla \phi_1|^\alpha + (c(x) + \bar{\lambda}) \phi_1^{\alpha+1} = 0 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

and of

$$\begin{cases} F(x, \nabla \phi_2, D^2 \phi_2) + b(x) \cdot \nabla \phi_2 |\nabla \phi_2|^\alpha + (c(x) + \bar{\lambda}) \phi_2 |\phi_2|^\alpha = 0 & \text{in } \Omega \\ \phi_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case $\alpha = 0$ these results are due to Busca, Esteban, Quaas [6], [14], and Ishii, Yoshimura [12], Quaas, Sirakov [15]. We wish to mention the recent work on multiplicity of solutions due to Sirakov [16] for related operators and the pioneering work of P.L. Lions [13].

When $\alpha \neq 0$ these problems have been studied in [3], [4]. In particular, for $\lambda < \bar{\lambda}$, we proved that the maximum principle holds and that there exists a solution u for the Dirichlet problem when the data f is negative in Ω and for zero boundary condition :

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + (c(x) + \lambda) u |u|^\alpha = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The scope of the present work is to enlarge these results to operators which are not homogeneous, to data that may change sign and non zero boundary condition.

Precisely, for $\lambda < \min\{\bar{\lambda}, \underline{\lambda}\}$, we shall study existence of solution, maximum principle and comparison principle, or lack of it, for Dirichlet problems of the following type:

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + (c(x) + \lambda) u |u|^\alpha + h(x, u) = f(x) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

depending on the choice of the function h .

The maximum principle will be proved under the assumption that $h(x, .)$ is non increasing and $h(x, 0) = 0$. The comparison principle holds if for all $x \in \Omega$ $t \mapsto \frac{-h(x, t)}{t^{\alpha+1}}$ is non decreasing on \mathbb{R}^+ . We shall also construct a counter-example to the comparison principle if this condition fails.

Finally, for continuous functions f and \mathcal{C}^2 functions g , we shall prove existence of solution if $h(x, u) = h_1(x, u) - h_2(x, u)$ with $h_i(., t) \in L^\infty$ for all t , $h_i(x, .)$ non-increasing and continuous for all $x \in \Omega$, $h_i(x, 0) = 0$ and if

$$\lim_{t \rightarrow \infty} \frac{h_2(x, t)}{t^{\alpha+1}} = 0.$$

2 Notations

In this section, we state the assumptions on the operators

$$G(x, \nabla u, D^2 u) := F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha$$

treated in this paper, and the notion of viscosity solution.

The operator F is continuous on $\Omega \times (\mathbb{R}^N)^* \times S$, where S denotes the space of symmetric matrices on \mathbb{R}^N .

The following hypothesis will be considered

(H1) $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \rightarrow \mathbb{R}$, and $\forall t \in \mathbb{R}^*$, $\mu \geq 0$, $F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$.

(H2) There exist $A \geq a > 0$ such that for all $x \in \Omega$, $p \neq 0$ and for all $(M, N) \in S^2$,
 $N \geq 0$

$$a|p|^\alpha \text{tr} N \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha \text{tr} N$$

(H3) There exists a continuous function $\tilde{\omega}$, $\tilde{\omega}(0) = 0$ such that for all x, y , $p \neq 0$,
 $\forall X \in S$

$$|F(x, p, X) - F(y, p, X)| \leq \tilde{\omega}(|x - y|) |p|^\alpha |X|.$$

(H4) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$
and $\zeta \in \mathbb{R}$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

where I is the identity matrix in \mathbb{R}^N , then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$F(x, \zeta(x - y), p, X) - F(y, \zeta(x - y), p, -Y) \leq \omega(\zeta|x - y|^2).$$

We shall suppose that $b : \Omega \mapsto \mathbb{R}^N$ is a continuous and bounded function satisfying:

(H5) -Either $\alpha < 0$ and b is Hölderian of exponent $1 + \alpha$,
- or $\alpha \geq 0$ and, for all x and y ,

$$\langle b(x) - b(y), x - y \rangle \leq 0$$

Let us recall what we mean by *viscosity solutions*, adapted to our context.

It is well known that, in dealing with viscosity respectively sub and super solutions, one works with

$$u^*(x) = \limsup_{y, |y-x| \leq r} u(y)$$

and

$$u_*(x) = \liminf_{y, |y-x| \leq r} u(y).$$

It is easy to see that $u_* \leq u \leq u^*$ and u^* is upper semicontinuous (USC) u_* is lower semicontinuous (LSC). See e.g. [9, 10].

Definition 1 Let Ω be a bounded domain in \mathbb{R}^N , and let g be a given continuous function in $\Omega \times \mathbb{R}$ then v , bounded on $\bar{\Omega}$, is called a viscosity super solution of $G(x, \nabla u, D^2u) = g(x, u)$ if for all $x_0 \in \Omega$ if

-Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $0 \leq g(x, c)$, for all $x \in B(x_0, \delta)$

-Or $\forall \varphi \in C^2(\Omega)$, such that $v_\star - \varphi$ has a local minimum on x_0 and $\nabla \varphi(x_0) \neq 0$, one has

$$G(x_0, \nabla \varphi(x_0), D^2\varphi(x_0)) \leq g(x_0, v_\star(x_0)). \quad (1)$$

Of course u is a viscosity sub solution if for all $x_0 \in \Omega$,

-Either there exists a ball $B(x_0, \delta)$, $\delta > 0$ on which $u = cte = c$ and $0 \geq g(x, c)$, for all $x \in B(x_0, \delta)$

-Or $\forall \varphi \in C^2(\Omega)$, such that $u^\star - \varphi$ has a local maximum on x_0 and $\nabla \varphi(x_0) \neq 0$, one has

$$G(x_0, \nabla \varphi(x_0), D^2\varphi(x_0)) \geq g(x_0, u^\star(x_0)). \quad (2)$$

A viscosity solution is a function which is both a super-solution and a sub-solution.

See e.g. [8] for a similar definition of viscosity solution of equations with singular operators.

For convenience we recall the definition of semi-jets given e.g. in [9]

$$\begin{aligned} J^{2,+}u(\bar{x}) = & \{(p, X) \in \mathbb{R}^N \times S, u(x) \leq u(\bar{x}) + \langle p, x - \bar{x} \rangle + \\ & + \frac{1}{2}\langle X(x - \bar{x}), x - \bar{x} \rangle + o(|x - \bar{x}|^2)\} \end{aligned}$$

and

$$\begin{aligned} J^{2,-}u(\bar{x}) = & \{(p, X) \in \mathbb{R}^N \times S, u(x) \geq u(\bar{x}) + \langle p, x - \bar{x} \rangle + \\ & + \frac{1}{2}\langle X(x - \bar{x}), x - \bar{x} \rangle + o(|x - \bar{x}|^2)\}. \end{aligned}$$

In the definition of viscosity solutions the test functions can be substituted by the elements of the semi-jets in the sense that in the definition above one can restrict to the functions ϕ defined by $\phi(x) = u(\bar{x}) + \langle p, x - \bar{x} \rangle + \frac{1}{2}\langle X(x - \bar{x}), x - \bar{x} \rangle$ with $(p, X) \in J^{2,-}u(\bar{x})$ when u is a super solution, and $(p, X) \in J^{2,+}u(\bar{x})$ when u is a sub solution.

For the convenience of the reader we now recall the properties obtained for $\lambda < \bar{\lambda}$ in [3].

Theorem 1 Let Ω be a bounded domain of \mathbb{R}^n . Suppose that F satisfies (H1), (H2), (H4), that b and c are continuous and b satisfies (H5). Suppose that $\tau < \bar{\lambda}$ and that u is a viscosity sub solution of

$$G(x, \nabla u, D^2u) + (c(x) + \tau)|u|^\alpha u \geq 0 \text{ in } \Omega$$

with $u \leq 0$ on the boundary of Ω , then $u \leq 0$ in Ω .

If $\tau < \underline{\lambda}$ and v is a super solution of

$$G(x, \nabla v, D^2v) + (c(x) + \tau)|v|^\alpha v \leq 0 \text{ in } \Omega$$

with $v \geq 0$ on the boundary of Ω then $v \geq 0$ in Ω .

Let us also recall the following comparison theorem.

Theorem 2 Suppose that F satisfies (H1), (H2), and (H4), that b and c are continuous and bounded and b satisfies (H5). Suppose that $\tau < \bar{\lambda}$, $f \leq 0$, f is upper semi-continuous and g is lower semi-continuous with $f \leq g$.

Suppose that there exist σ upper semi continuous , and v non-negative and lower semi continuous , satisfying

$$\begin{aligned} G(x, \nabla v, D^2 v) + (c(x) + \tau)v^{1+\alpha} &\leq f \quad \text{in } \Omega \\ G(x, \nabla \sigma, D^2 \sigma) + (c(x) + \tau)|\sigma|^\alpha \sigma &\geq g \quad \text{in } \Omega \\ \sigma &\leq v \quad \text{on } \partial\Omega \end{aligned}$$

Then $\sigma \leq v$ in Ω in each of these two cases:

- 1) If $v > 0$ on $\overline{\Omega}$ and either $f < 0$ in Ω , or $g(\bar{x}) > 0$ on every point \bar{x} such that $f(\bar{x}) = 0$,
- 2) If $v > 0$ in Ω , $f < 0$ and $f < g$ on $\overline{\Omega}$

We recall some of the properties of the distance function for bounded C^2 set.

Remark 1 In all the paper we shall consider that Ω is a bounded C^2 domain. In particular we shall use several times the fact that this implies that the distance to the boundary:

$$d(x, \partial\Omega) := d(x) := \inf\{|x - y|, y \in \partial\Omega\}$$

satisfies the following properties:

1. d is Lipschitz continuous
2. There exists $\delta > 0$ such that in $\Omega_\delta = \{x \in \Omega \text{ such that } d(x) \leq \delta\}$, d is $C^{1,1}$.
3. d is semi-concave, i.e. there exists $C_1 > 0$ such that $d(x) - C_1|x|^2$ is concave and this implies $J^{2,+}d(x) \neq \emptyset$ for any $x \in \Omega$.
4. If $J^{2,-}d(x) \neq \emptyset$, d is differentiable at x and $|\nabla d(x)| = 1$.

3 An existence's result for C^2 boundary data

We assume that F satisfies the assumptions enumerated in the previous section.

Let g be given in $W^{2,\infty}(\partial\Omega)$, and $f \in L^\infty$. We denote by b a bounded and continuous function which satisfies (H5). We introduce some functions h_1 and h_2 :

(H6) For $i = 1$ and $i = 2$ let $h_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that $h_i(., t) \in L^\infty$ for all t , $h_i(x, .)$ is non-increasing and continuous for all $x \in \Omega$, $h_i(x, 0) = 0$ and

$$\lim_{t \rightarrow \infty} \frac{h_2(x, t)}{t^{\alpha+1}} = 0.$$

Our main existence's result is the following theorem:

Theorem 3 Suppose that $\lambda < \lambda_1 = \inf\{\bar{\lambda}, \underline{\lambda}\}$ and suppose that $h_1(x, t)$ and $h_2(x, t)$ satisfy (H6) then, for $g \in W^{2,\infty}(\partial\Omega)$ and $f \in \mathcal{C}(\overline{\Omega})$, there exists a solution of

$$\begin{cases} G(x, \nabla u, D^2 u) + (c(x) + \lambda)u|u|^\alpha + h_1(x, u) = f(x) + h_2(x, u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

In particular, Theorem 3 implies that, for $\beta_1(x) \geq 0$ and $\beta_2(x) \geq 0$, for any $q_1 > 0$ and for $q_2 < \alpha$, there exists u solution of

$$\begin{cases} G(x, \nabla u, D^2 u) + (c(x) + \lambda)|u|^\alpha u - \beta_1(x)|u|^{q_1}u = \\ \quad f(x) - \beta_2(x)|u|^{q_2}u & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Remark 2 The previous existence's result still holds when $\underline{\lambda} < \lambda < \bar{\lambda}$ if $f \leq 0$ and $g \geq 0$. The proof proceeds as the one of Theorem 3 using the Remark 4 which is stated after Theorem 5 and the fact that $u \equiv 0$ is a subsolution.

A symmetric result holds for $\underline{\lambda} > \lambda > \bar{\lambda}$.

The proof of this Theorem requires several step :

The first step is given by:

Proposition 1 Suppose that g is in $W^{2,\infty}(\partial\Omega)$, that $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $h(x, .)$ is non increasing and continuous and f is in L^∞ . Then there exists u a viscosity solution of

$$\begin{cases} G(x, \nabla u, D^2 u) + h(x, u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (3)$$

To prove this proposition, it is enough to construct a sub and a super solution of (3) and then apply Perron's method (see [4, 10]). This is the purpose of the following two Propositions 2, 3.

Then, in Theorem 4, we will prove a Hölder's estimate which also gives a compactness result. And then the proof of Theorem 3 will be done through a recursive argument.

Proposition 2 Suppose that g is in $W^{2,\infty}(\partial\Omega)$, that $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $h(x, .)$ is non increasing and continuous and $m \in \mathbb{R}^+$. Then there exists u a viscosity subsolution of

$$\begin{cases} G(x, \nabla u, D^2 u) + h(x, u) \geq m & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Proof of Proposition 2. We denote by g a $\mathcal{C}^2 \cap W^{2,\infty}(\Omega)$ function which equals g on $\partial\Omega$.

Let us note first that it is sufficient to construct a subsolution v when $h \equiv 0$, as long as it satisfies $v \leq g$.

Indeed, suppose that we have constructed a subsolution of the equation

$$G(x, \nabla v, D^2 v) \geq m' := m - h(x, g)$$

this will imply, since h is decreasing, that

$$G(x, \nabla v, D^2v) + h(x, v) \geq m - h(x, g) + h(x, v) \geq m.$$

We assume now that $h \equiv 0$.

Let us consider, for L and k to be chosen later, the function $v(x) = g(x) + L((1+d(x))^{-k} - 1)$. Let D be an upper bound for d on Ω , and C_1 be a positive constant as in 3 of Remark 1.

We choose $k > 1$ large enough in order that $a(k+1) > 2(A+a)NC_1(1+D)$ and $k+1 > \frac{2^{\alpha+2}}{a}|b|_\infty(1+D)^{\alpha+2}$. Next we choose L such that

$$Lk \geq \sup \left\{ 2|\nabla g|_\infty(1+D)^{k+1}, 2\frac{(A+a)}{a}(1+D)^{k+2}|D^2g|_\infty, \left(\frac{16m'(1+D)^{k(\alpha+1)+\alpha+2}}{a(k+1)}\right)^{\frac{1}{1+\alpha}} \right\}.$$

Let us recall that since d is semi-concave for every $x \in \Omega$ $J^{2,+}d(x) \neq \emptyset$.

To prove that v is a subsolution, let $x_0 \in \Omega$ be any point such that there exists a test function $\varphi \in C^2$ satisfying

$$g(x) + \left(\frac{1}{(1+d)^k} - 1\right)(x) \leq \varphi(x), \quad g(x_0) + \left(\frac{1}{(1+d)^k} - 1\right)(x_0) = \varphi(x_0).$$

Then $\frac{1}{(\varphi(x)-g(x)+1)^{\frac{1}{k}}} - 1 \leq d(x)$ and $J^{2,-}d(x_0) \neq \emptyset$.

Since $J^{2,+}d(x_0) \neq \emptyset$ this implies that d is differentiable in x_0 and then $\nabla d(x_0) = \nabla \left(\frac{1}{(\varphi-g+1)^{\frac{1}{k}}} - 1\right)(x_0)$ which of course implies that

$$\nabla v(x_0) = \nabla \varphi(x_0) = (\nabla g - Lk(1+d)^{-k-1}\nabla d)(x_0).$$

Hence, for simplicity, we shall use directly v instead of the test function. Moreover since d is semiconcave, one has, for the constant C_1 defined in Remark 1, for all x and \bar{x}

$$d(x) - C_1|x|^2 - d(\bar{x}) + C_1|\bar{x}|^2 \leq \nabla d(\bar{x}) - C_1\bar{x}.(x - \bar{x}).$$

This implies that for every $(p, X) \in J^{2,-}d(\bar{x})$, $X \leq C_1I$.

Observe that in particular, with the above choice of L and k one has

$$|\nabla v| \geq \frac{Lk(1+D)^{-k-1}}{2}.$$

We are now in a position to compute D^2v :

$$D^2v = D^2g + Lk(k+1)(1+d)^{-k-2}\nabla d \otimes \nabla d - Lk(1+d)^{-k-1}D^2d.$$

Recalling that $\nabla d \otimes \nabla d \geq 0$ and $|\nabla d| = 1$, we get

$$\begin{aligned} \mathcal{M}_{a,A}^-(D^2v) &\geq Lk \left(a \frac{(k+1)}{(1+d)^{k+2}} - \frac{(A+a)C_1N}{(1+d)^{k+1}} \right) - (A+a)|D^2g|_\infty \\ &\geq \frac{Lka(k+1)}{2(1+d)^{k+2}} - (A+a)|D^2g|_\infty \\ &\geq \frac{Lk(k+1)a}{4(1+d)^{k+2}}. \end{aligned}$$

As a consequence

$$\begin{aligned} |\nabla v|^\alpha \mathcal{M}_{a,A}^-(D^2v) + b(x) \cdot \nabla v |\nabla v|^\alpha &\geq \frac{(Lk)^{\alpha+1}}{(1+D)^{k(\alpha+1)+\alpha+2}} \left(\frac{a(k+1)}{2^{\alpha+2}} - |b|_\infty (1+D)^{\alpha+2} \right) \\ &\geq \frac{(Lk)^{\alpha+1}a(k+1)}{2^{\alpha+3}(1+d)^{k(\alpha+1)+\alpha+2}} \\ &\geq 2m'. \end{aligned}$$

We then obtain the required inequality,

$$\begin{aligned} G(x, \nabla v, D^2v) &\geq \mathcal{M}_{a,A}^-(D^2v) |\nabla v|^\alpha - |b|_\infty |\nabla v|^{\alpha+1} \\ &\geq m' \end{aligned}$$

This ends the proof.

Proposition 3 Suppose that $m > 0$ that $g \in W^{2,\infty}(\partial\Omega)$ and $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $h(x, \cdot)$ is non increasing and continuous. Then there exists a supersolution u of

$$\begin{cases} F(x, \nabla u, D^2u) + b(x) \cdot \nabla u |\nabla u|^\alpha + h(x, u) \leq -m & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Remark 3 In the following we shall denote by $\bar{u} = S(g, -m)$ a supersolution as in Proposition 3 and by $\underline{u} = S(g, m)$ a subsolution as in Proposition 2.

Proof of Proposition 3. We still denote by g a $C^2 \cap W^{2,\infty}$ function on $\bar{\Omega}$ which equals g on the boundary. As in the previous proposition it is sufficient to prove the result when $h \equiv 0$ as long as $\varphi \geq g$.

We choose $\varphi(x) := g(x) + L(1 - (1 + d(x))^{-k})$, with L and K appropriate constants to be chosen later.

To prove that φ is a supersolution, either $J^{2,-}\varphi(x_0) = \emptyset$ or not, and then there exists $\psi(x) \leq g(x) + L(1 - (1 + d)^{-k})(x)$ touching φ at x_0 . This implies that

$1 - (1 - g(x) + \varphi)(x)^{-k}(x) \geq d(x)$, and $J^{2,-}d(x) \neq \emptyset$. Then $J^{2,+}d(x_0)$ and $J^{2,-}d(x_0)$ are non empty, and d is differentiable on x_0 . We shall use in the following ∇d for the computations below. As in the previous proof, k is chosen large enough in order that

$$(k+1) \geq \sup \left(\frac{2(A+a)C_1N(1+D)}{a}, \frac{|b|_\infty 4(1+D)}{a} \right).$$

Now we can choose L such that

$$L \geq \sup \left(\left(\frac{4m'}{a(k+1)} \right)^{\frac{1}{\alpha+1}} \frac{2(1+d)^{k+1}}{k}, \frac{2(A+a)|D^2g|_\infty(1+D)^{k+2}}{ka(k+1)}, \frac{2|\nabla g|_\infty}{k(1+D)^{k+1}} \right).$$

The computation of the gradient gives

$$\nabla \varphi = \nabla g + Lk(1+d)^{-k-1} \nabla d.$$

and by the previous assumptions $|\nabla \varphi| \geq \frac{L}{2}k(1+d)^{-k-1}$.

While

$$D^2\varphi = D^2g + \frac{Lk}{(1+d)^{k+1}} \left[-\frac{(k+1)}{(1+d)} \nabla d \otimes \nabla d + dD^2d \right].$$

We then have that

$$\mathcal{M}_{a,A}^+(\nabla \nabla \varphi) \leq (A+a)|D^2g|_\infty + \frac{Lk}{(1+d)^{k+1}} \left(-a \frac{(k+1)}{(1+d)} + (A+a)dC_1N \right).$$

Since

$$a \frac{(k+1)}{2(1+d)} \geq (A+a)dC_1N,$$

one gets,

$$\mathcal{M}_{a,A}^+(\nabla \nabla \varphi) \leq (A+a)|D^2g| - \frac{Lk(k+1)a}{2(1+d)^{k+2}} \leq -\frac{Lka(k+1)}{4(1+d)^{k+2}}.$$

We can finish the computation, and get;

$$|\nabla \varphi|^\alpha \mathcal{M}_{a,A}(\nabla \nabla \varphi) + b(x) \cdot \nabla \varphi |\nabla \varphi|^\alpha \leq \left(\frac{Lk}{2(1+d)^{k+1}} \right)^{\alpha+1} \left(-\frac{a(k+1)}{2(1+d)} + |b|_\infty \right)$$

Hence, with our choice of the constants, we have obtained

$$G(x, \nabla \varphi, D^2 \varphi) \leq -m'.$$

This ends the proof.

We now give regularity results for solutions with boundary data in $W^{2,\infty}$.

This immediately implies a compacity result for fixed data $W^{2,\infty}(\partial\Omega)$, extending in that way the results in [4].

Theorem 4 Suppose that $g \in W^{2,\infty}(\partial\Omega)$ then every solution of

$$\begin{cases} G(x, \nabla u, D^2 u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

satisfies: For any $\gamma \in (0, 1)$ there exists C depending on γ , $|g|_{W^{2,\infty}(\partial\Omega)}$ and $|f|_\infty$

$$|u(x) - u(y)| \leq C|x - y|^\gamma$$

for all x and y in Ω .

Corollary 1 Suppose that $g_n \rightarrow g$ in $W^{2,\infty}(\partial\Omega)$, that f_n is a bounded sequence of bounded functions and that u_n are solutions of

$$\begin{cases} G(x, \nabla u_n, D^2 u_n) = f_n & \text{in } \Omega \\ u_n = g_n & \text{on } \partial\Omega \end{cases}$$

Suppose that (u_n) is bounded in L^∞ , then u_n is uniformly Hölderian, and the sequence is relatively compact in $\mathcal{C}(\overline{\Omega})$.

Proof of Theorem 4 . We proceed similarly to [4, 11]. Let us recall that the proof has two steps. In the first one Hölder's regularity is proved near the boundary. And then it is proved in the interior, through a typical viscosity argument. We only give the details of the first part, since the second part proceeds as in [4].

Hence we define for a given positive δ

$$\Omega_\delta := \{(x, y) \in \Omega^2 \text{ such that } |x - y| \leq \delta\}$$

and the first step consists in proving that on $\partial\Omega_\delta$ there exists $C > 0$ such that

$$u(x) - u(y) \leq C|x - y|^\gamma.$$

If $C > \frac{2|u|_\infty}{\delta}$ then the inequality is true for $|x - y| = \delta$ so we should only prove it for $(x, y) \in \Omega \times \partial\Omega$ and similarly for $(x, y) \in \partial\Omega \times \Omega$.

We shall still denote by g a $C^2 \cap W^{2,\infty}(\Omega)$ extension of g to Ω .

Using the Propositions 2 and 3 we know that there exist L, k , which depend only on universal constants and on the C^2 norm of g , such that $g(x) + L((1+d)^{-k} - 1)$ is a subsolution and $g(x) + L(1 - (1+d)^{-k})$ is a super solution.

Using the comparison theorem in [2], one gets that

$$g(x) + L((1+d)^{-k} - 1) \leq u \leq g(x) + L(1 - (1+d)^{-k})$$

Finally there exist C and C_g some Lipschitz constant of g such that on $\partial\Omega_\delta$, if $d(x) = \delta$ and $y \in \partial\Omega$, one has

$$u(x) - u(y) = u(x) - g(y) \geq g(x) - g(y) - Cd(x)^\gamma \geq -C_g|x - y| - C|x - y|^\gamma$$

and

$$u(x) - u(y) \leq u(x) - g(y) \leq g(x) - g(y) + C(d(x))^\gamma \leq C_g|x - y| + C|x - y|^\gamma.$$

The rest of the proof proceeds as in [4].

We pass to the second step, which treats the case where f and g are constant, with opposite sign.

Theorem 5 Suppose that we have the same hypothesis as in Theorem 3 in particular for $\lambda < \lambda_1$. Then for all $g \in \mathbb{R}^+$, $f \in \mathbb{R}^+$ there exist $\bar{u} \geq 0$ and $\underline{u} \leq 0$ respectively solutions of

$$\begin{cases} G(x, \nabla \bar{u}, D^2 \bar{u}) + (c(x) + \lambda)|\bar{u}|^\alpha \bar{u} + h_1(x, \bar{u}) = -f + h_2(x, \bar{u}) & \text{in } \Omega \\ \bar{u} = g & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} G(x, \nabla \underline{u}, D^2 \underline{u}) + (c(x) + \lambda)|\underline{u}|^\alpha \underline{u} + h_1(x, \underline{u}) = f + h_2(x, \underline{u}) & \text{in } \Omega \\ \underline{u} = -g & \text{on } \partial\Omega. \end{cases}$$

Remark 4 If $\underline{\lambda} < \lambda < \bar{\lambda}$ the same existence of \bar{u} holds when $f \leq 0$ and $g \geq 0$ and symmetrically if $\underline{\lambda} > \lambda > \bar{\lambda}$ for \underline{u} .

Proof of Theorem 5. We consider the first case, the other being symmetric.

Let u_n be the sequence defined as the positive solution of

$$\begin{cases} G(x, \nabla u_n, D^2 u_n)(c + \lambda - 2|c + \lambda|_\infty)u_n^{1+\alpha} + h_1(x, u_n) = \\ -f - 2|\lambda + c|_\infty u_{n-1}^{1+\alpha} + h_2(x, u_{n-1}) & \text{in } \Omega \\ u_n = g, & \text{on } \partial\Omega \end{cases}$$

for $n \geq 1$ and $u_0 = 0$, which exists by Proposition 1.

By construction and the comparison principle, u_n is increasing and hence $u_n \geq 0$. Let us prove that it is bounded. Indeed if not, $|u_n|_\infty \rightarrow +\infty$ and the function $w_n = \frac{u_n}{|u_n|_\infty}$ solves

$$\begin{aligned} & G(x, \nabla w_n, D^2 w_n) + (c + \lambda - 2|c + \lambda|_\infty) w_n^{1+\alpha} + \frac{h_1(x, u_n)}{|u_n|^{\alpha+1}} \\ &= \frac{f}{|u_n|_\infty^{1+\alpha}} - 2|\lambda + c|_\infty \frac{u_{n-1}^{1+\alpha}}{|u_n|_\infty^{1+\alpha}} + \frac{h_2(x, u_{n-1})}{|u_n|_\infty^{\alpha+1}}. \end{aligned}$$

By hypothesis on h_1 and h_2 , $h_1(x, u_n) \leq 0$ and

$$\frac{h_2(x, u_{n-1})}{|u_n|_\infty^{\alpha+1}} \geq \frac{h_2(x, |u_n|_\infty)}{|u_n|_\infty^{\alpha+1}} \rightarrow 0.$$

Extracting subsequences one has for some increasing sequence $\sigma(n)$

$$\frac{|u_{\sigma(n)-1}|_\infty}{|u_{\sigma(n)}|_\infty} \rightarrow k \leq 1$$

and

$$w_{\sigma(n)} \rightarrow w \geq 0$$

with $|w|_\infty = 1$ and $w = 0$ on the boundary of Ω . Moreover, using the compactness result in Corollary 1, one gets that w satisfies

$$G(x, \nabla w, D^2 w) + (c + \lambda - 2|c + \lambda|_\infty(1 - k^{\alpha+1})) w^{1+\alpha} \geq 0$$

with $w = 0$ on the boundary. Since $\lambda - 2|c + \lambda|_\infty(1 - k^{\alpha+1}) < \lambda_1$, by the maximum principle we get that $w \leq 0$. But $w \geq 0$ and hence $w \equiv 0$, which contradicts $|w|_\infty = 1$.

We have proved that (u_n) is bounded. Since (u_n) is monotone, by the compactness result of Corollary 1, it converges to some u which is the required solution. This ends the proof.

Proof of Theorem 3. We need to construct a solution of

$$\begin{cases} G(x, \nabla u, D^2 u) + (c + \lambda)|u|^\alpha u + h_1(x, u) = f + h_2(x, u) & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

with f in L^∞ and g which is C^2 .

We denote respectively $\underline{u} = S(2|f|_\infty, -|g|_\infty)$ $\bar{u} = S(-2|f|_\infty, |g|_\infty)$ the solutions obtained in the previous theorem.

We define a sequence u_n by the recursive process :

$$\begin{cases} G(x, \nabla u_n, D^2 u_n) + ((c + \lambda) - 2|c + \lambda|_\infty)|u_n|^\alpha u_n + h_1(x, u_n) \\ \quad = f - 2|c + \lambda|_\infty |u_{n-1}|^\alpha u_{n-1} + h_2(x, u_{n-1}) & \text{in } \Omega \\ u_n = g, & \text{on } \partial\Omega \end{cases}$$

initializing with $u_0 = \underline{u}$. We know that the sequence is well defined by Proposition 1 with $h(x, u) = ((c + \lambda) - |c + \lambda|_\infty)|u|^\alpha u + h_1(x, u)$.

Let us prove that

$$\underline{u} \leq u_n \leq \bar{u}.$$

Let us note that since $\underline{u} \leq 0 \leq \bar{u}$ it is equivalent to prove that $u_n^+ \leq \bar{u}$ and $u_n^- \leq -\underline{u}$.

Suppose that we know that $\underline{u} \leq u_{n-1} \leq \bar{u}$ and let us prove that $\underline{u} \leq u_n \leq \bar{u}$.

One has

$$\begin{aligned} & G(x, \nabla u_n, D^2 u_n) + (c + \lambda - 2|c + \lambda|_\infty) u_n |u_n|^\alpha + h_1(x, u_n) \\ &= f - 2|\lambda + c|_\infty |u_{n-1}|^\alpha u_{n-1} + h_2(x, u_{n-1}) \\ &\leq |f|_\infty + 2|\lambda + c|_\infty |u_{n-1}^-|^{1+\alpha} + h_2(x, -u_{n-1}^-) \\ &\leq |f|_\infty - 2|\lambda + c|_\infty |\underline{u}|^\alpha \underline{u} + h_2(x, \underline{u}) \\ &< G(x, \nabla \underline{u}, D^2 \underline{u}) + (c + \lambda - 2|c + \lambda|_\infty) \underline{u} |\underline{u}|^\alpha + h_1(x, \underline{u}). \end{aligned}$$

Since on the boundary $u_n \geq \underline{u}$ one gets that $u_n \geq \underline{u}$.

Now

$$\begin{aligned} & G(x, \nabla u_n, D^2 u_n) + (c + \lambda - 2|c + \lambda|_\infty) u_n |u_n|^\alpha + h_1(x, u_n) = \\ &= f - 2|\lambda + c|_\infty |u_{n-1}|^\alpha u_{n-1} + h_2(x, u_{n-1}) \\ &\geq -|f|_\infty - 2|\lambda + c|_\infty |u_{n-1}^+|^{1+\alpha} + h_2(x, u_{n-1}^+) \\ &\geq -|f|_\infty - 2|\lambda + c|_\infty \bar{u}^{\alpha+1} + h_2(x, \bar{u}) \\ &> G(x, \nabla \bar{u}, D^2 \bar{u}) + (c + \lambda - 2|c + \lambda|_\infty) \bar{u}^{\alpha+1} + h_1(x, \bar{u}). \end{aligned}$$

Since on the boundary $u_n = g \leq \bar{u}$ one gets that

$$u_n \leq \bar{u}.$$

It is sufficient now to invoke the compacity result to see that the sequence u_n is relatively compact in $C(\bar{\Omega})$. One gets, passing to the limit, that for a subsequence $u_n \rightarrow u$ which satisfies

$$\begin{cases} G(x, \nabla u, D^2 u) + (c(x) + \lambda)|u|^\alpha u + h_1(x, u) = f + h_2(x, u) & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

This ends the proof.

3.1 The case $f(x) \equiv 0$.

Proposition 4 Suppose that for all x and u , h satisfies $h(x, u)u \leq 0$. Then for $\lambda < \lambda_1$ the only solution of

$$\begin{cases} G(x, \nabla u, D^2 u) + (\lambda + c)|u|^\alpha u + h(x, u) = 0 & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

is $u \equiv 0$.

Proof Suppose that there exists a non zero solution. Then either $\Omega^+ = \{x, u(x) > 0\} \neq \emptyset$ or $\Omega^- = \{x, u(x) < 0\} \neq \emptyset$. Without loss of generality we can suppose that $\Omega^+ \neq \emptyset$ then

$$G(x, \nabla u, D^2 u) + (\lambda + c)|u|^\alpha u = -h(x, u) \geq 0$$

in Ω^+ and $u = 0$ on the boundary of Ω^+ . As seen in [4], $\bar{\lambda}(\Omega^+) \geq \bar{\lambda}(\Omega) > \lambda$, then by the maximum principle $u \leq 0$ in Ω^+ which is a contradiction. This ends the proof.

We consider here β some continuous and bounded function and $q < \alpha$.

We are interested in the existence of non trivial solutions of

$$\begin{cases} G(x, \nabla u, D^2 u) + (\lambda + c)|u|^\alpha u + \beta|u|^q u = 0 & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (4)$$

Theorem 6 Suppose that $\lambda < \lambda_1$. Then, if $\beta = \beta^+ - \beta^-$ with $\beta^+ \not\equiv 0$ there exists a non trivial solution of (4).

Proof of Theorem 6. First we suppose that $\beta \geq 0$, β not identically zero. We begin to construct a subsolution with the aid of some eigenfunction. Let $\phi > 0$ be such that

$$\begin{cases} G(x, \nabla \phi, D^2 \phi) + (\bar{\lambda} + c)\phi^{1+\alpha} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

with $|\phi|_\infty = 1$. If m is small enough in order that

$$(m\phi)^{\alpha-q}|\beta|_\infty < \frac{\bar{\lambda} - \lambda}{2}$$

then $m\phi$ is a subsolution of (4).

Let u_n be defined in a recursive way by

$$\begin{cases} G(x, \nabla u_n, D^2 u_n) + (c + \lambda - 2|c + \lambda|_\infty)|u_n|^\alpha u_n = \\ \quad -2|\lambda + c|_\infty |u_{n-1}|^\alpha u_{n-1} - \beta|u_{n-1}|^q u_{n-1}, & \text{in } \Omega \\ u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

with $u_0 = m\phi$. The solutions u_n are well defined by Proposition 1. We begin by proving that $u_1 \geq m\phi$.

$$\begin{aligned} & G(x, \nabla u_1, D^2 u_1) + (c + \lambda - 2|c + \lambda|_\infty)|u_1|^\alpha u_1 \\ &= -2|\lambda + c|_\infty u_0^{\alpha+1} - \beta u_0^{q+1} \\ &\leq -2|\lambda + c|_\infty m^{1+\alpha} \phi^{\alpha+1} - m^{q+1} \beta \phi^{q+1} \\ &\leq G(x, \nabla(m\phi), D^2(m\phi)) + (c + \lambda - 2|c + \lambda|_\infty)(m\phi)^{\alpha+1} \end{aligned}$$

this implies, by the comparison principle, that $u_1 \geq m\phi$.

The same reasoning establishes that $u_n \geq u_{n-1}$. Suppose that $|u_n|_\infty \rightarrow +\infty$, then, defining $w_n = \frac{u_n}{|u_n|_\infty}$, one gets that

$$\begin{cases} G(x, \nabla w_n D^2 w_n) + (c + \lambda - 2|c + \lambda|_\infty)|w_n|^\alpha w_n = \\ \quad -2|\lambda + c|_\infty |w_{n-1}|^\alpha w_{n-1} \frac{|u_{n-1}|_\infty^{1+\alpha}}{|u_n|_\infty^{1+\alpha}} - \beta \frac{|u_{n-1}|_\infty^q}{|u_n|_\infty^{1+\alpha}} |w_{n-1}|^q w_{n-1}, & \text{in } \Omega \\ w_n = 0, & \text{on } \partial\Omega. \end{cases}$$

By the compactness result, w_n converges, up to a subsequence, to some non negative function w which is such that $|w|_\infty = 1$, and for some $k \leq 1$ it is a solution of

$$\begin{cases} G(x, \nabla w, D^2 w) + (c + \lambda - 2|c + \lambda|_\infty(1 - k^{\alpha+1}))|w|^{1+\alpha} w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, since $\lambda - 2|c| + c(1 - k^{1+\alpha}) < \bar{\lambda}$, one gets that $w = 0$, a contradiction. Finally the sequence u_n is increasing and bounded and by the compactness result obtained in Corollary 1, it converges towards u which is a solution of (4). Since $u \geq m\phi$, it is non trivial.

We now consider the case where β changes sign. We begin to construct a subsolution.

Let $\Omega^+ = \{x \in \Omega \text{ such that } \beta(x) > 0\}$ which by hypothesis is not empty. The previous case ensures that there exists a non negative solution, not identically zero, denoted \tilde{u}_0 such that

$$\begin{cases} G(x, \nabla \tilde{u}_0, D^2 \tilde{u}_0) + (c + \lambda)(\tilde{u}_0)^{\alpha+1} = -\beta^+(\tilde{u}_0)^{q+1}, & \text{in } \Omega^+ \\ \tilde{u}_0 = 0, & \text{on } \partial\Omega^+. \end{cases}$$

We shall denote by u_0 the extension:

$$u_0 = \begin{cases} \tilde{u}_0 & \text{in } \Omega^+ \\ 0 & \text{in } \Omega \setminus \Omega^+. \end{cases}$$

It is immediate to see that u_0 is a nonnegative viscosity subsolution of

$$\begin{cases} G(x, \nabla u_0, D^2 u_0) + (c + \lambda)(u_0)^{\alpha+1} + \beta(u_0)^{q+1} \geq 0, & \text{in } \Omega \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

Let u_n be defined in a recursive way as

$$\begin{cases} G(x, \nabla u_n, D^2 u_n) + (c + \lambda - 2|c + \lambda|_\infty)|u_n|^\alpha u_n - \beta^-|u_n|^q u_n = \\ -2|\lambda + c|_\infty |u_{n-1}|^\alpha u_{n-1} - \beta^+|u_{n-1}|^q u_{n-1} & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Again, this sequence is well defined by Proposition 1. We have $u_n \geq u_0$, and even more precisely that $u_n \geq u_{n-1}$.

We claim that u_n is bounded. Suppose by contradiction that $|u_n|_\infty \rightarrow +\infty$, then defining $w_n = \frac{u_n}{|u_n|_\infty}$, one easily obtains, as in the previous case, that w_n converges, up to a subsequence, to some function w which satisfies for some $k \leq 1$:

$$G(x, \nabla w, D^2 w) + (c + \lambda - 2|c + \lambda|_\infty(1 - k^{1+\alpha}))w^{1+\alpha} = 0$$

and is zero on the boundary. One gets a contradiction with the maximum principle and then the sequence (u_n) is bounded. Extracting from it a subsequence, using Corollary 1 and passing to the limit one gets that u is a solution. Since $u \geq u_0$ which is not identically zero, we get the result.

4 Maximum and comparison principles

Theorem 7 Suppose that F satisfies (H1), (H2), (H4), that b and c are continuous and b satisfies (H5). Suppose that h is a continuous function such that $h(x, \cdot)$ is non increasing, $h(x, 0) = 0$. Suppose that $\tau < \bar{\lambda}$ and u is a viscosity sub solution of

$$G(x, \nabla u, D^2 u) + h(x, u) + (\tau + c(x))|u|^\alpha u \geq 0 \quad \text{in } \Omega.$$

If $u \leq 0$ on the boundary of Ω , then $u \leq 0$ in Ω .

Remark: Similarly it is possible to prove that if $\tau < \lambda$ and v is a super solution of

$$G(x, \nabla v, D^2 v) + (\tau + c(x))|v|^\alpha v + h(x, v) \leq 0 \quad \text{in } \Omega.$$

If $v \geq 0$ on the boundary of Ω then $v \geq 0$ in Ω .

Corollary 2 Suppose that c' is some continuous function, $c' < c + \bar{\lambda}_c$. Then the maximum principle holds for the equation

$$G(x, \nabla v, D^2 v) + c'|v|^\alpha v = 0$$

or equivalently if v is some solution of

$$G(x, \nabla v, D^2 v) + c'|v|^\alpha v \geq 0$$

and $v \leq 0$ on the boundary, then $v \leq 0$ in Ω

For the proof of the corollary it is sufficient to use the previous result with $h(x, u) = (c' - (c + \lambda))|u|^\alpha u$, where $\lambda < \bar{\lambda}$ and $c' \leq c + \lambda$.

Before starting the proof let us remind two results proved in [4]:

Proposition 5 Suppose that F satisfies (H1) and (H2), and that b is bounded.

Let u be uppersemicontinuous subsolution of

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha \geq -m & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for some constant $m \geq 0$. Then there exists $\delta > 0$ and some constant C_3 that depends only on the structural data such that, for x satisfying $d(x) \leq \delta$, u satisfies

$$u(x) \leq C_3 d(x).$$

Proposition 6 (Hopf) Let v be a viscosity continuous super solution of

$$F(x, \nabla v, D^2 v) + b(x) \cdot \nabla v |\nabla v|^\alpha + c(x)|v|^\alpha v \leq 0.$$

Suppose that v is positive in a neighborhood of $x_o \in \partial\Omega$ and $v(x_o) = 0$ then there exist $C > 0$ and $\delta > 0$ such that

$$v(x) \geq C|x - x_o|$$

for $|x - x_o| \leq \delta$.

Proof of Theorem 7.

Let $\lambda \in]\tau, \bar{\lambda}[$, and let v be a super solution of

$$G(x, \nabla v, D^2 v) + (\lambda + c(x))v^{\alpha+1} \leq 0,$$

satisfying $v > 0$ in Ω , which exists by definition of $\bar{\lambda}$.

We assume by contradiction that $\sup u(x) > 0$ in Ω . We first want to prove that $\sup \frac{u}{v} < +\infty$.

Let $d(x) = d(x, \partial\Omega)$ the distance from the boundary of Ω .

Clearly from Proposition 5 and 6 there exists $\delta > 0$ such that $u(x) \leq Cd(x)$ and $v(x) \geq C'd(x)$ for $d(x) \leq \delta$, for some constants C and C' .

In the interior we just use the fact that $v \geq C'\delta > 0$ in $\Omega_\delta = \{x : d(x) \geq \delta\}$ and we can conclude that $\frac{u}{v}$ is bounded in Ω .

We now define $\gamma' = \sup_{x \in \Omega} \frac{u}{v}$ achieved on some point \bar{y} and $w(x) = \gamma v(x)$, where $0 < \gamma < \gamma'$, and γ is sufficiently close to γ' in order that $\frac{\lambda - \tau \left(\frac{\gamma'}{\gamma}\right)^{1+\alpha}}{\left(\frac{\gamma'}{\gamma}\right)^{1+\alpha} - 1} \geq 2|c|_\infty$.

Furthermore by definition of the supremum there exists $x \in \Omega$ such that $\frac{u(x)}{v(x)} \geq \gamma$.

Clearly, by homogeneity, $G(x, \nabla w, D^2w) + (c(x) + \lambda)w^{1+\alpha} \leq 0$.

The supremum of $u - w$ is strictly positive, and it is necessarily achieved inside Ω , -say on \bar{x} - since on the boundary $u - w \leq 0$.

Let us note that

$$(u - w)(\bar{x}) \geq (u - w)(\bar{y}) = (\gamma' - \gamma)v(\bar{y})$$

and

$$(u - w)(\bar{x}) \leq (\gamma' - \gamma)v(\bar{x})$$

which implies that

$$\frac{\gamma'}{\gamma}w(\bar{x}) \geq u(\bar{x}).$$

We consider, for $j \in \mathbb{N}$ and for some $q > \max(2, \frac{\alpha+2}{\alpha+1})$:

$$\psi_j(x, y) = u(x) - w(y) - \frac{j}{q}|x - y|^q.$$

Since $\sup(u - w) > 0$, the supremum of ψ_j is achieved on $(x_j, y_j) \in \Omega^2$. It is classical that for j large enough, ψ_j achieves its positive maximum on some couple $(x_j, y_j) \in \Omega^2$ such that

- 1) $x_j \neq y_j$ for j large enough,
- 2) $(x_j, y_j) \rightarrow (\bar{x}, \bar{x})$ which is a maximum point for $u - w$ and \bar{x} is an interior point
- 3) $j|x_j - y_j|^q \rightarrow 0$,
- 4) there exist X_j and Y_j in S such that

$$(j|x_j - y_j|^{q-2}(x_j - y_j), X_j) \in J^{2,+}u(x_j)$$

and

$$(j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \in J^{2,-}w(y_j).$$

Furthermore

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq j \begin{pmatrix} D_j & -D_j \\ -D_j & D_j \end{pmatrix} \leq 2^{q-2}jq(q-1)|x_j - y_j|^{q-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

with

$$D_j = 2^{q-3}q|x_j - y_j|^{q-2}(I + \frac{(q-2)}{|x_j - y_j|^2}(x_j - y_j) \otimes (x_j - y_j)).$$

The proof of these facts proceeds similarly to the one given in [4].

Condition (H4) implies that

$$F(x_j, j(x_j - y_j)|x_j - y_j|^{q-2}, X_j) - F(y_j, j(x_j - y_j)|x_j - y_j|^{q-2}, -Y_j) \leq \omega(j|x_j - y_j|^q).$$

Then, using the above inequality, the properties of the sequence (x_j, y_j) , the condition on b -with C_b below being either the Hölder constant or 0-, and the homogeneity condition (H1) one obtains

$$\begin{aligned} -h(x_j, u(x_j)) - (\tau + c(x_j))u(x_j)^{1+\alpha} &\leq G(x_j, j(x_j - y_j)|x_j - y_j|^{q-2}, X_j) \\ &\leq G(y_j, j(x_j - y_j)|x_j - y_j|^{q-2}, -Y_j) \\ &+ \omega(j|x_j - y_j|^q) + C_b j^{1+\alpha} |x_j - y_j|^{q(1+\alpha)} + o(1) \\ &\leq -(\lambda + c(y_j))w(y_j)^{1+\alpha} + o(1). \end{aligned}$$

By passing to the limit when j goes to infinity, since c is continuous one gets

$$-(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} - h(\bar{x}, u(\bar{x})) \leq -(\lambda + c(\bar{x}))w(\bar{x})^{1+\alpha}$$

Suppose first that $c(\bar{x}) + \lambda > 0$ then using the inequality on w and the fact that $-h(\bar{x}, u(\bar{x})) \geq 0$, one gets

$$-(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} \leq -(\lambda + c(\bar{x}))w(\bar{x})^{1+\alpha}$$

and then

$$-(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} \leq -(\lambda + c(\bar{x})) \left(\frac{\gamma}{\gamma'} u(\bar{x}) \right)^{1+\alpha},$$

which is a contradiction with the assumption on γ and γ' .

If $(\lambda + c(\bar{x})) = 0$ then $\tau < \lambda$ implies that $-(\tau + c(\bar{x})) > 0$ and then

$$0 < -(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} \leq -(\lambda + c(\bar{x})) \left(\frac{\gamma}{\gamma'} u(\bar{x}) \right)^{1+\alpha} = 0$$

a contradiction.

Suppose finally that $\lambda + c(\bar{x}) < 0$ then, using

$$w(\bar{x}) \leq u(\bar{x}) + \left(\frac{\gamma}{\gamma'} - 1 \right) u(\bar{y}) \leq u(\bar{x})$$

we get

$$-(c(\bar{x}) + \tau)u(\bar{x})^{1+\alpha} \leq -(c(\bar{x}) + \lambda)u(\bar{x})^{1+\alpha}.$$

This implies implies that

$$(\lambda - \tau)u(\bar{x})^{1+\alpha} \leq 0$$

a contradiction. This ends the proof.

We now prove a comparison result.

Theorem 8 Suppose that F satisfies (H1), (H2), and (H4), that b and c are continuous and bounded and b satisfies (H5). Suppose that $\tau < \bar{\lambda}$, $f_1 \leq 0$, f_1 is upper semi-continuous and f_2 is lower semi-continuous with $f_1 \leq f_2$. Suppose that h is such that, for all $x \in \Omega$, $t \mapsto \frac{-h(x,t)}{t^{\alpha+1}}$ is non decreasing on \mathbb{R}^+ . Suppose that there exist σ , and v non-negative, with

$$\begin{aligned} G(x, \nabla v, D^2 v) + (c(x) + \tau)v^{1+\alpha} + h(x, v) &\leq f_1 \text{ in } \Omega \\ G(x, \nabla \sigma, D^2 \sigma) + (c(x) + \tau)|\sigma|^\alpha \sigma + h(x, \sigma) &\geq f_2 \text{ in } \Omega \\ \sigma &\leq v \text{ on } \partial\Omega \end{aligned}$$

Then $\sigma \leq v$ in Ω in each of these two cases

- 1) If $v > 0$ on $\overline{\Omega}$ and either $f_1 < 0$ in Ω , or $f_2(\bar{x}) > 0$ on every point \bar{x} such that $f_1(\bar{x}) = 0$,
- 2) If $v > 0$ in Ω , $f_1 < 0$ and $f_1 < f_2$ on $\overline{\Omega}$

Remark 5 Of course a similar comparison principle can be proved for $\tau < \underline{\lambda}$ and non positive solutions.

Corollary 3 Suppose that $c + \lambda > 0$ and $\lambda < \bar{\lambda}$. Let u and v be two solutions of

$$G(x, \nabla u, D^2 u) + (c + \lambda)|u|^\alpha u = 0 \quad \text{in } \Omega.$$

- 1) If $u \geq v$ on $\partial\Omega$
Then $u \geq v$ in Ω .
- 2) If $u > v > 0$ on $\partial\Omega$ then $u > v$ in Ω .

In particular this implies that if $g > 0$ the solution of

$$\begin{cases} G(x, \nabla u, D^2 u) + (c + \lambda)|u|^\alpha u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is unique.

Proof of Corollary 3 Let $u_\epsilon = u - \epsilon$ and $v_\epsilon = \frac{v}{1+\gamma\epsilon}$ with ϵ and γ chosen conveniently in order that $u_\epsilon \geq v_\epsilon$ on $\partial\Omega$. They satisfy

$$G(x, \nabla u_\epsilon, D^2 u_\epsilon) + (\lambda + c)(u_\epsilon)^{1+\alpha} < 0$$

and

$$G(x, \nabla v_\epsilon, D^2 v_\epsilon) + (\lambda + c)(v_\epsilon)^{1+\alpha} = 0.$$

Hence we are in the hypothesis of the comparison theorem and $u_\epsilon \geq v_\epsilon$ in Ω . Passing to the limit one gets the result.

We now treat the strict comparison principle. Since $u > v > 0$ there exists ϵ such that $u \geq (1 + \epsilon)v$ on the boundary. Since $v(1 + \epsilon)$ is still a solution by homogeneity, one gets by the first part of the corollary that $u \geq (1 + \epsilon)v$ in Ω and then $u > v$.

Proof of Theorem 8.

We act as in the proof of Theorem 3.6 in [2].

- 1) We assume first that $v > 0$ on $\overline{\Omega}$.

Suppose by contradiction that $\sigma > v$ somewhere in Ω . The supremum of the function $\frac{\sigma}{v}$ on $\partial\Omega$ is less than 1 since $\sigma \leq v$ on $\partial\Omega$ and $v > 0$ on $\partial\Omega$, then its supremum is achieved inside Ω . Let \bar{x} be a point such that

$$1 < \frac{\sigma(\bar{x})}{v(\bar{x})} = \sup_{x \in \bar{\Omega}} \frac{\sigma(x)}{v(x)}.$$

We define

$$\psi_j(x, y) = \frac{\sigma(x)}{v(y)} - \frac{j}{qv(y)} |x - y|^q.$$

For j large enough, this function achieves its maximum which is greater than 1, on some couple $(x_j, y_j) \in \Omega^2$. It is easy to see that this sequence converges to (\bar{x}, \bar{x}) , a maximum point for $\frac{\sigma}{v}$. We prove as in [4] that x_j, y_j can be chosen such that $x_j \neq y_j$ for j large enough.

Moreover there exist X_j and Y_j such that

$$\left(j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)} \right) \in J^{2,+}\sigma(x_j)$$

and

$$\left(j|x_j - y_j|^{q-2}(x_j - y_j) \frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j} \right) \in J^{2,-}v(y_j)$$

where $\beta_j = \sigma(x_j) - \frac{j}{q}|x_j - y_j|^q$
and

$$F(x_j, j|x_j - y_j|^{q-2}(x_j - y_j), X_j) - F(y_j, j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \leq \omega(v(y_j)j|x_j - y_j|^q).$$

We can use the fact that σ and v are respectively sub and super solution to obtain:

$$\begin{aligned} f_2(x_j) - (\tau + c(x_j))\sigma(x_j)^{1+\alpha} - h(x_j, \sigma(x_j)) &\leq G(x_j, j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)}) \\ &\leq \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} \left\{ F(y_j, j|x_j - y_j|^{q-2}(x_j - y_j) \frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}) \right. \\ &\quad \left. + \omega(jv(y_j)|x_j - y_j|^q) \right\} + \\ &\quad + b(x_j) \cdot j^{1+\alpha} |x_j - y_j|^{(q-1)(1+\alpha)-1} (x_j - y_j) \\ &\leq \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} G(y_j, j|x_j - y_j|^{q-2}(x_j - y_j) \frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}) \\ &\quad + \frac{\omega(v(y_j)j|x_j - y_j|^q)}{v(y_j)^{1+\alpha}} + C(j|x_j - y_j|^{q+1+\alpha}) \\ &\leq (-\tau - c(y_j))\beta_j^{1+\alpha} - \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} h(y_j, v(y_j)) + \\ &\quad \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} f_1(y_j) + o(1). \end{aligned}$$

Passing to the limit, since c is continuous, we get:

$$f_2(\bar{x}) \leq \left(\frac{\sigma(\bar{x})}{v(\bar{x})} \right)^{\alpha+1} f_1(\bar{x}) + h(\bar{x}, \sigma(\bar{x})) - \left(\frac{\sigma(\bar{x})}{v(\bar{x})} \right)^{1+\alpha} h(\bar{x}, v(\bar{x})) \leq \left(\frac{\sigma(\bar{x})}{v(\bar{x})} \right)^{\alpha+1} f_1(\bar{x})$$

By the hypothesis on h .

$$f_1(\bar{x}) \left[1 - \left(\frac{\sigma(\bar{x})}{v(\bar{x})} \right)^{\alpha+1} \right] \leq f_1(\bar{x}) - f_2(\bar{x}) \leq 0.$$

This contradicts the hypothesis on f and g .

2) Suppose that $v > 0$ and let $m > 0$ be such that $f_1 < f_2 - m$ in $\bar{\Omega}$.

Using continuity there exists $\varepsilon_o > 0$ such that for any $\varepsilon \leq \varepsilon_o$:

$$|c + \lambda|_\infty ((v + \varepsilon)^{\alpha+1} - v^{\alpha+1}) + |h(x, v + \varepsilon) - h(x, v)| < \frac{m}{2}.$$

Then $w := v + \varepsilon$ is a solution of

$$G(x, \nabla w, D^2 w) + (c(x) + \lambda)w^{1+\alpha} + h(x, w) \leq f_1(x) + \frac{m}{2} < f_2(x)$$

furthermore $w \geq \sigma$ on $\partial\Omega$. And we can conclude using the first part that $v + \varepsilon \geq \sigma$. Letting ε go to zero we obtain the required result.

Remark: The condition on the increasing behavior of h in Theorem 8 is somehow optimal in the sense that it is possible to construct a counter example when $\frac{h(x,t)}{t^{\alpha+1}}$ is non increasing :

Proposition 7 *Let $q < \alpha$ and $\beta > 0$, $\lambda > 0$. There exists $\epsilon_2 > 0$ and $M > 0$ such that for any continuous function f satisfying $0 > f(x) > -\epsilon_2$ there exist two solutions u and v of*

$$\begin{cases} G(x, \nabla u, D^2 u) - \beta u^{1+q} + \lambda u^{1+\alpha} = f(x) & \text{in } \Omega \\ u = M & \text{on } \partial\Omega \end{cases}$$

with $u \leq M \leq v$, $u \not\equiv M$ and $v \not\equiv M$.

Proof.

Let $k(t) = -\beta t^{q+1} + \lambda t^{1+\alpha}$, defined on \mathbb{R}^+ . Let $M > 0$ and M' be defined as

$$k'(M) = 0$$

$$k(M') = 0.$$

On has $M < M'$. Let $\epsilon_2 > 0$ such that

$$k(M) = -\epsilon_2.$$

Let $m(x) \in (M, M')$ such that $k(m(x)) = f(x) > -\epsilon_2$.

We define first a sequence (u_n) for $n \geq 1$:

$$\begin{cases} G(x, \nabla u_n, D^2 u_n) - \beta u_n^{1+q} = f - \lambda u_{n-1}^{1+\alpha} & \text{in } \Omega \\ u_n = M & \text{on } \partial\Omega; \end{cases}$$

initializing with $u_0 = 0$. One easily has $u_n \geq 0$ and $u_n \leq M$. Indeed if $u_{n-1} \leq M$

$$\begin{aligned} G(x, \nabla u_n, D^2 u_n) - \beta u_n^{1+q} &\geq k(m(x)) - \lambda M^{1+\alpha} \\ &= k(m(x)) - \beta M^{1+q} - k(M) \\ &> f(x) - \beta M^{1+q} - \epsilon_2 > -\beta M^{1+q}. \end{aligned}$$

On the other hand, by definition, M is a super solution and then

$$u_n \leq M$$

since on the boundary $u_n = M$.

In the same manner one can check that u_n is increasing , and by passing to the limit one gets a solution. This solution is between 0 and M and cannot be equal to M .

We define next a sequence of solutions of

$$\begin{cases} G(x, \nabla v_n, D^2 v_n) - \beta v_n^{1+q} = k(m(x)) - \lambda v_{n-1}^{1+\alpha} & \text{in } \Omega \\ v_n = M & \text{on } \partial\Omega \end{cases}$$

initializing with $v_0 = M'$.

Let us prove that $v_n \geq M$:

$$\begin{aligned} G(x, \nabla v_1, D^2 v_1) - \beta v_1^{1+q} &= k(m(x)) - \lambda(M')^{1+\alpha} \\ &= -\beta m(x)^{1+q} + \lambda m(x)^{1+\alpha} - \lambda(M')^{1+\alpha} \\ &\leq -\beta m(x)^{1+q} \leq -\beta M^{1+q}. \end{aligned}$$

by the comparison principle $v_1 \geq M$. In the same manner we can prove the induction step and the result holds for any n .

We now prove that $v_1 \leq M'$:

$$\begin{aligned} G(x, \nabla v_1, D^2 v_1) - \beta v_1^{1+q} &= k(m(x)) - \lambda(M')^{1+\alpha} \\ &= k(m(x)) - k(M') - \beta(M')^{1+\alpha} \geq -\beta(M')^{1+\alpha}, \end{aligned}$$

then $v_1 \leq M'$. In the same manner, one can prove the induction step and that the sequence is decreasing.

The limit is a solution which is between M and M' and cannot be equal to M . This ends the proof.

References

- [1] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*. Comm. Pure Appl. Math. **47** (1994), no. 1, 47–92.
- [2] I. Birindelli, F. Demengel, Comparison principle and Liouville type results for singular fully nonlinear operators, Ann. Fac. Sci. Toulouse Math, (6)13 (2004), N.2, 261-287., 2004.
- [3] I. Birindelli, F. Demengel, *Eigenvalue and Maximum principle for fully nonlinear singular operators* Advances in Partial Diff. Equations **11** n.1 (2006), 91-119.

- [4] I. Birindelli, F. Demengel, *Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators* to appear in Comm. Pure and Applied Analysis.
- [5] I. Birindelli, F. Demengel, J. Wigniolle, *Strict maximum principle*, to appear in the Proceedings of Workshop on Second Order Subelliptic Equations and Applications- Cortona, (2003).
- [6] J. Busca,M.J. Esteban, A. Quaas *Nonlinear eigenvalues and bifurcation problems for Pucci's operator* Annales de l'Institut H. Poincaré, Analyse non linéaire **22** (2005) 187–206.
- [7] L. Caffarelli, X. Cabré, Fully-nonlinear equations. Colloquium Publications 43, American Mathematical Society, Providence, RI,1995.
- [8] Y. G. Chen, Y. Giga, S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*. J. Differential Geom. **33** (1991), no. 3, 749–786.
- [9] M.G. Crandall, H. Ishii, P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations* , Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.
- [10] H. Ishii, *Viscosity solutions of non-linear partial differential equations* , Sugaku Expositions vol 9 , (1996).
- [11] H. Ishii, P.L. Lions, *Viscosity solutions of Fully- Nonlinear Second Order Elliptic Partial Differential Equations*, J. Differential Equations **83** (1990), no. 1, 26–78.
- [12] H. Ishii, Y. Yoshimura, *Demi-eigen values for uniformly elliptic Isaacs operators* , preprint.
- [13] P.-L. Lions, *Bifurcation and optimal stochastic control*, Nonlinear Anal. **7** (1983), no. 2, 177–207.
- [14] A. Quaas, *Existence of positive solutions to a “semilinear” equation involving the Pucci's operators in a convex domain*, Differential Integral Equations **17** (2004), no. 5–6, 481–494.
- [15] A. Quaas, B. Sirakov, *On the principal eigenvalues and the Dirichlet problem for fully nonlinear operators*. C. R. Math. Acad. Sci. Paris **342** (2006), no. 2, 115–118.
- [16] B. Sirakov, *Non uniqueness for the Dirichlet problem for fully nonlinear elliptic operators* , preprint.